The complete factorizations of $f(x)$ over the set of real numbers and over the set of complex numbers are shown below.

$$x^4 - 5x^3 + 4x^2 + 2x - 8 = (x+1)(x-4)(x^2 - 2x + 2)$$
$$= (x+1)(x-4)[x - (1+i)][x - (1-i)]$$

Both products can be verified by multiplication. ■

An expression that has *even* degree of 4 or greater may have only complex roots. Zeros and factors of such functions and expressions may be difficult to find, and the techniques illustrated in this chapter are of little use. However, functions of *odd* degree must have at least one real zero, and the corresponding expression must have at least one real linear factor. Therefore, a cubic expression can easily be approximately factored by estimating one zero, which yields the real linear factor, using synthetic division to determine the quadratic factor, and then using the quadratic formula to estimate the remaining two zeros.

Complete 10 problems

Exercises 4.6 (I'd do all 14 though)

In Exercises 1–6, determine if $g(x)$ is a factor of $f(x)$ without using synthetic or long division.

1. $f(x) = x^{10} + x^8 \quad g(x) = x - 1$
2. $f(x) = x^6 - 10 \quad g(x) = x - 2$
3. $f(x) = 3x^4 - 6x^3 + 2x - 1 \quad g(x) = x + 1$
4. $f(x) = x^5 - 3x^2 + 2x - 1 \quad g(x) = x - 2$
5. $f(x) = x^3 - 2x^2 + 5x - 4 \quad g(x) = x + 2$
6. $f(x) = 10x^{75} - 8x^{65} + 6x^{45} + 4x^{32} - 2x^{15} + 5$
   $g(x) = x - 1$

In Exercises 7–10, list the zeros of the polynomial and state the multiplicity of each zero.

7. $f(x) = x^{54}\left(x + \frac{4}{5}\right)$
8. $g(x) = 3\left(x + \frac{1}{6}\right)\left(x - \frac{1}{5}\right)\left(x + \frac{1}{4}\right)$
9. $h(x) = 2x^{15}(x - \pi)^{14}[x - (\pi + 1)]^{13}$
10. $k(x) = (x - \sqrt{7})^7(x - \sqrt{5})^5(2x - 1)$

In Exercises 11–22, find all the zeros of f in the complex number system; then write $f(x)$ as a product of linear factors.

11. $f(x) = x^2 - 2x + 5$
12. $f(x) = x^2 - 4x + 13$
13. $f(x) = 3x^2 + 2x + 7$
14. $f(x) = 3x^2 - 5x + 2$
15. $f(x) = x^3 - 27$ *Hint:* Factor first.
16. $f(x) = x^3 + 125$
17. $f(x) = x^3 + 8$
18. $f(x) = x^6 - 64$
    Hint: Let $u = x^3$ and factor $u^2 - 64$ first.
19. $f(x) = x^4 - 1$
20. $f(x) = x^4 - x^2 - 6$
21. $f(x) = x^4 - 3x^2 - 10$
22. $f(x) = 2x^4 - 7x^2 - 4$

In Exercises 23–44, find a polynomial $f(x)$ with real coefficients that satisfies the given conditions. Some of the problems have many correct answers.

23. degree 3; only zeros are $1, 7, -4$
24. degree 3; only zeros are 1 and -1
25. degree 6; only zeros are $1, 2, \pi$
26. degree 5; only zero is 2
27. degree 3; zeros $-3, 0, 4$; $f(5) = 80$
28. degree 3; zeros $-1, \frac{1}{2}, 2$; $f(0) = 2$
29. zeros include $2 + i$ and $2 - i$

30. zeros include $1 + 3i$ and $1 - 3i$

31. zeros include 2 and $2 + i$

32. zeros include 3 and $4i - 1$

33. zeros include $-3, 1 - i, 1 + 2i$

34. zeros include $1, 2 + i, 3i - 1$

35. degree 2; zeros $1 + 2i$ and $1 - 2i$

36. degree 4; zeros $3i$ and $-3i$, each of multiplicity 2

37. degree 4; only zeros are $4, 3 + i$, and $3 - i$

38. degree 5; zeros 2 of multiplicity 3, i, and $-i$

39. degree 6; zeros 0 of multiplicity 3 and $3, 1 + i, 1 - i$, each of multiplicity 1

40. degree 6; zeros include i of multiplicity 2 and 3

41. degree 2; zeros include $1 + i$; $f(0) = 6$

42. degree 2; zeros include $3 + i$; $f(2) = 3$

43. degree 3; zeros include i and 1; $f(-1) = 8$

44. degree 3; zeros include $2 + 3i$ and -2; $f(2) = -3$

In Exercises 45–48, find a polynomial with complex coefficients that satisfies the given conditions.

45. degree 2; zeros i and $1 - 2i$

46. degree 2; zeros $2i$ and $1 + i$

47. degree 3; zeros $3, i$, and $2 - i$

48. degree 4; zeros $\sqrt{2}, -\sqrt{2}, 1 + i$, and $1 - i$

In Exercises 49–56, one zero of the polynomial is given; find all the zeros. Then write in factored form

49. $x^3 - 2x^2 - 2x - 3$; zero 3

50. $x^3 + x^2 + x + 1$; zero i

51. $x^4 + 3x^3 + 3x^2 + 3x + 2$; zero i

52. $x^4 - x^3 - 5x^2 - x - 6$; zero i

53. $x^4 - 2x^3 + 5x^2 - 8x + 4$; zero 1 of multiplicity 2

54. $x^4 - 6x^3 + 29x^2 - 76x + 68$; zero 2 of multiplicity 2

55. $x^4 - 4x^3 + 6x^2 - 4x + 5$; zero $2 - i$

56. $x^4 - 5x^3 + 10x^2 - 20x + 24$; zero $2i$

57. Let $z = a + bi$ and $w = c + di$ be complex numbers (a, b, c, d are real numbers). Prove the given equality by computing each side and comparing the results.

a. $\overline{z + w} = \overline{z} + \overline{w}$ (The left side says: "First find $z + w$ and then take the conjugate." The right side says: "First take the conjugates of z and w and then add.")

b. $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$

58. Let $g(x)$ and $h(x)$ be polynomials of degree n and assume that there are $n + 1$ numbers $c_1, c_2, \ldots, c_n, c_{n+1}$ such that

$$g(c_i) = h(c_i) \quad \text{for every } i.$$

Prove that $g(x) = h(x)$. *Hint:* Show that each c_i is a root of $f(x) = g(x) - h(x)$. If $f(x)$ is nonzero, what is its largest possible degree? To avoid a contradiction, conclude that $f(x) = 0$.

59. Suppose $f(x) = ax^3 + bx^2 + cx + d$ has real coefficients and z is a complex zero of f.

a. Use Exercise 57 and the fact that $\overline{r} = r$, when r is a real number, to show that

$$\overline{f(z)} = \overline{az^3 + bz^2 + cz + d} = a\overline{z}^3 + b\overline{z}^2 + c\overline{z} + d = f(\overline{z}).$$

b. Conclude that $\overline{z}$ is also a zero of f. *Note:* $f(\overline{z}) = \overline{f(z)} = \overline{0} = 0$.

60. Let $f(x)$ be a polynomial with real coefficients and z a complex zero of f. Prove that the conjugate $\overline{z}$ is also a zero of f. *Hint:* Exercise 58 is the case when $f(x)$ has degree 3; the proof in the general case is similar.

61. Use the Factorization over the Real Numbers statement to show that every polynomial with real coefficients and odd degree must have at least one real root.

62. Give an example of a polynomial $f(x)$ with complex, nonreal coefficients and a complex number z such that z is a zero of f but its conjugate is not. Therefore, the conclusion of the Conjugate Roots Theorem may be false if $f(x)$ doesn't have real coefficients.

Exercises 4.2

When asked to find the zeros of a polynomial, find exact zeros whenever possible and approximate the other zeros.

In Exercises 1–12, find all the rational zeros of the polynomial.

1. $x^3 + 3x^2 - x - 3$
2. $x^3 - x^2 - 3x + 3$
3. $x^3 + 5x^2 - x - 5$
4. $3x^3 + 8x^2 - x - 20$
5. $2x^5 + 5x^4 - 11x^3 + 4x^2$ *Hint:* The Rational Zero Test can only be used on polynomials with nonzero constant terms. Factor $f(x)$ as a product of a power of x and a polynomial $g(x)$ with nonzero constant term. Then use the Rational Zero Test on $g(x)$.
6. $2x^6 - 3x^5 - 7x^4 - 6x^3$
7. $\frac{1}{12}x^3 - \frac{1}{12}x^2 - \frac{2}{3}x + 1$ *Hint:* The Rational Zero Test can only be used on polynomials with integer coefficients. Note that $f(x)$ and $12f(x)$ have the same zeros. (Why?)
8. $\frac{2}{3}x^4 + \frac{1}{2}x^3 - \frac{5}{4}x^2 - x - \frac{1}{6}$
9. $\frac{1}{3}x^4 - x^3 - x^2 + \frac{13}{3}x - 2$
10. $\frac{1}{3}x^7 - \frac{1}{2}x^6 - \frac{1}{6}x^5 + \frac{1}{6}x^4$
11. $0.1x^3 - 1.9x + 3$
12. $0.05x^3 + 0.45x^2 - 0.4x + 1$

In Exercises 13–18, factor the polynomial as a product of linear factors and a factor $g(x)$ such that $g(x)$ is either a constant or a polynomial that has no rational zeros.

13. $2x^3 - 4x^2 + x - 2$
14. $6x^3 - 5x^2 + 3x - 1$
15. $x^6 + 2x^5 + 3x^4 + 6x^3$
16. $x^5 - 2x^4 + 2x^3 - 3x + 2$
17. $x^5 - 4x^4 + 8x^3 - 14x^2 + 15x - 6$
18. $x^5 + 4x^3 + x^2 + 6x$

In Exercises 19–22, use the Bounds Test to find lower and upper bounds for the real zeros of the polynomial.

19. $x^3 + 2x^2 - 7x + 20$
20. $x^3 - 15x^2 - 16x + 12$
21. $-x^5 - 5x^4 + 9x^3 + 18x^2 - 68x + 176$ *Hint:* The Bounds Test applies only to polynomials with a positive leading coefficient. The polynomial $f(x)$ has the same zeros as $-f(x)$. Why?
22. $-0.002x^3 - 5x^2 + 8x - 3$

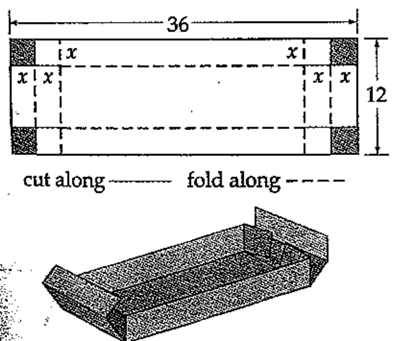
In Exercises 23–36, find all ~~real~~ zeros of the polynomial.

23. $2x^3 - 5x^2 + x + 2$
24. $t^4 - t^3 + 2t^2 - 4t - 8$
25. $6x^3 - 11x^2 + 6x - 1$
26. $z^3 + z^2 + 2z + 2$
27. $x^4 + x^3 - 19x^2 + 32x - 12$ (Hint: Try 2 twice
28. $3x^5 + 2x^4 - 7x^3 + 2x^2$
29. $2x^5 - x^4 - 10x^3 + 5x^2 + 12x - 6$
30. $x^5 - x^3 + x$
31. $x^6 - 4x^5 - 5x^4 - 9x^2 + 36x + 45$
32. $x^5 + 3x^4 - 4x^3 - 11x^2 - 3x + 2$
33. $3x^4 + 2x^3 - 4x^2 + 4x - 1$
34. $x^5 + 8x^4 + 20x^3 + 9x^2 - 27x - 27$
35. $x^4 - 48x^3 - 101x^2 + 49x + 50$
36. $3x^7 + 8x^6 - 13x^5 - 36x^4 - 10x^3 + 21x^2 + 41x + 10$
37. **a.** Show that $\sqrt{2}$ is an irrational number. *Hint:* $\sqrt{2}$ is a zero of $x^2 - 2$. Does this polynomial have any rational zeros?
    b. Show that $\sqrt{3}$ is irrational.
38. Graph $f(x) = 0.001x^3 - 0.199x^2 - 0.23x + 6$ in the standard viewing window.
    a. How many zeros does $f(x)$ appear to have? Without changing the viewing window, explain why $f(x)$ must have an additional zero. *Hint:* Each zero corresponds to a factor of $f(x)$. What does the rest of the factorization consist of?
    b. Find all the zeros of $f(x)$.
39. According to the FBI, the number of people murdered each year per 100,000 population can be approximated by the polynomial function $f(x) = 0.0011x^4 - 0.0233x^3 + 0.1144x^2 + 0.0126x +$ 8.1104 $(0 \leq x \leq 10)$, where $x = 0$ corresponds to 1987.

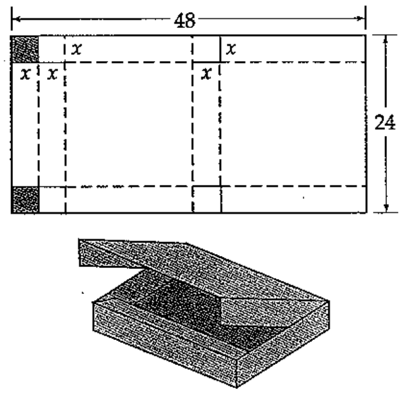
a. What was the murder rate in 1990?
b. In what year was the rate 8 people per 100,000?
c. In what year was the rate the highest?

40. During the first 150 hours of an experiment, the growth rate of a bacteria population at time t hours is $g(t) = -0.0003t^3 + 0.04t^2 + 0.3t + 0.2$ bacteria per hour.
a. What is the growth rate at 50 hours? at 100 hours?
b. What is the growth rate at 145 hours? What does this mean?
c. At what time is the growth rate 0?
d. At what time is the growth rate -50 bacteria per hour?
e. Approximately at what time does the highest growth rate occur?

41. An open-top reinforced box is to be made from a 12-by-36-inch piece of cardboard by cutting along the marked lines, discarding the shaded pieces, and folding as shown in the figure. If the box must be less than 2.5 inches high, what size squares should be cut from the corners in order for the box to have a volume of 448 cubic inches?



42. A box with a lid is to be made from a 48-by-24-inch piece of cardboard by cutting and folding, as shown in the figure. If the box must be at least 6 inches high, what size squares should be cut from the two corners in order for the box to have a volume of 1000 cubic inches?



43. In a sealed chamber where the temperature varies, the instantaneous rate of change of temperature with respect to time over an 11-day period is given by $F(t) = 0.0035t^4 - 0.4t^2 - 0.2t + 6$, where time is measured in days and temperature in degrees Fahrenheit (so that rate of change is in degrees per day).
a. At what rate is the temperature changing at the beginning of the period ($t = 0$)? at the end of the period ($t = 11$)?
b. When is the temperature increasing at a rate of 4°F per day?
c. When is the temperature decreasing at a rate of 3°F per day?
d. When is the temperature decreasing at the fastest rate?

44. *Critical Thinking*
a. If c is a zero of

$$f(x) = 5x^4 - 4x^3 + 3x^2 - 4x + 5,$$

show that $\frac{1}{c}$ is also a zero.
b. Do part a with $f(x)$ replaced by $g(x)$.

$$g(x) = 2x^6 + 3x^5 + 4x^4 - 5x^3 + 4x^2 + 3x + 2$$

c. Let

$$f(x) = a_{12}x^{12} + a_{11}x^{11} + \cdots + a_2x^2 + a_1x + a_0.$$

If c is a zero of f, what conditions must the coefficients a_i satisfy so that $\frac{1}{c}$ is also a zero?